foot of the altitude of $\triangle ABM$ from $M$ and let $A - M_1 - B$. Prove that then $\overline{MA} > \overline{MB}$ if and only if $\overline{M_1A} > \overline{M_1B}$.

**8.** If $M$ is the midpoint of $\overline{BC}$ then $\overline{AM}$ is called a **median** of $\triangle ABC$. Consider $\triangle ABC$ such that $\overline{AB} < \overline{AC}$. Let $E$, $D$ and $H$ denote the points in which bisector of angle, median and altitude from $A$ intersect line $\overleftrightarrow{BC}$, respectively. Show that (a) $\measuredangle AEB < \measuredangle AEC$; (b) $\overline{BE} < \overline{CE}$; (c) we have $H - E - D$.

**9.** (a.) Prove that in a neutral geometry if $\triangle ABC$ is isosceles with base $\overline{BC}$ then the following are collinear: (i) the median from $A$; (ii) the bisector of $\measuredangle A$; (iii) the altitude from $A$; (iv) the perpendicular bisector of $\overline{BC}$. (b.) Conversely, in a neutral geometry prove that if any two of (i)-(iv) are collinear then the triangle is isosceles (six different cases).

**10.** Show that the conclusion of the Pythagorean Theorem is not valid in the Poincaré Plane by considering $\triangle ABC$ with $A(2,1)$, $B(0,\sqrt{5})$, and $C(0,1)$. Thus the Pythagorean Theorem does not hold in every neutral geometry.

**Theorem** In a neutral geometry, if $\overrightarrow{BD}$ is the bisector of $\measuredangle ABC$ and if $E$ and $F$ are the feet of the perpendiculars from $D$ to $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$ then $\overline{DE} \cong \overline{DF}$.

**11.** Prove the above Theorem. [Th 6.4.7, p 148]

## 20 Circles and Their Tangent Lines

**Definition**. **(circle with center $C$ and radius $r$, chord, diameter, radius segment).** If $C$ is a point in a metric geometry $(\mathcal{S}, \mathcal{L}, d)$ and if $r > 0$, then

$$\mathcal{C} = \mathcal{C}_r(C) = \{P \in \mathcal{S} \,|\, PC = r\}$$

is a circle with center $C$ and radius $r$. If $A$ and $B$ are distinct points of $\mathcal{C}$ then $\overline{AB}$ is a chord of $\mathcal{C}$. If the center $C$ is a point on the chord $\overline{AB}$, then $\overline{AB}$ is a diameter of $\mathcal{C}$. For any $Q \in \mathcal{C}$, $\overline{CQ}$ is called a radius segment of $\mathcal{C}$.

**1.** Find and sketch the circle of radius 1 with center $(0,0)$ in the Euclidean Plane and in the Taxicab Plane. [Ex 6.5.1, p150]

**2.** Consider $\{\mathbb{R}^2, \mathcal{L}_E\}$ with the max distance $d_s$ (recall $d_s(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ where $P(x_1, y_1)$ and $Q(x_2, y_2)$ denote two points in $\mathbb{R}^2$). Sketch the circle $\mathcal{C}_1((0,0))$.

**3.** Show that $\mathcal{A} = \{(x,y) \in \mathbb{H} \,|\, x^2 + (y-5)^2 = 16\}$ is the Poincaré circle $\mathcal{C}$ with center $(0,3)$ and radius $\ln 3$. [Ex 6.5.2, p151]

Our first result tells us that in a neutral geometry the center and radius of a circle are determined by any three points on the circle.

**Theorem**. In a neutral geometry, let $\mathcal{C}_1 = \mathcal{C}_r(C)$ and $\mathcal{C}_2 = \mathcal{C}_s(D)$. If $\mathcal{C}_1 \cap \mathcal{C}_2$ contains at least three points, then $C = D$ and $r = s$. Thus, three points of a circle in a neutral geometry uniquely determine that circle.

**4.** Prove the above Theorem. [Th 6.5.3, p152]

**Corollary**. For any circle in a neutral geometry, the perpendicular bisector of any chord contains the center.

**5.** If $\overline{AB}$ is a chord of a circle in a neutral geometry but is not a diameter, prove that the line through the midpoint of $\overline{AB}$ and the center of the circle is perpendicular to $\overline{AB}$.

**6.** Prove that a line in a neutral geometry intersects a circle at most twice.

**Definition**. **(interior, exterior).** Let $\mathcal{C}$ be the circle with center $C$ and radius $r$. The interior of $\mathcal{C}$ is the set $\text{int}(\mathcal{C}) = \{P \in \mathcal{S} \,|\, CP < r\}$. The exterior of $\mathcal{C}$ is the set $\text{ext}(\mathcal{C}) = \{P \in \mathcal{S} \,|\, CP > r\}$.

**Theorem**. If $\mathcal{C}$ is a circle in a neutral geometry then $\text{int}(\mathcal{C})$ is convex.

**7.** Prove the above Theorem. [Th 6.5.5, p153]

**Definition**. **(tangent, point of tangency).** In a metric geometry, a line $\ell$ is a tangent to the circle $\mathcal{C}$ if $\ell \cap \mathcal{C}$ contains exactly one point (which is called the point of tangency). $\ell$ is called a secant of the circle $\mathcal{C}$ if $\ell \cap \mathcal{C}$ has exactly two points.

**8.** In the Taxicab Plane prove that for the circle $\mathcal{C} = \mathcal{C}_1((0,0))$: (a). There are exactly four points at which a tangent to $\mathcal{C}$ exists. (b). At each point in part (a) there are infinitely many tangent lines.

**Theorem**. In a neutral geometry, let $\mathcal{C}$ be a circle with center $C$ and let $Q \in \mathcal{C}$. If $t$ is a line through $Q$, then $t$ is tangent to $\mathcal{C}$ if and only if $t$ is perpendicular to the radius segment $\overline{CQ}$.

**9.** Prove the above Theorem. [Th 6.5.6, p154]

**Corollary**. **(Existence and Uniqueness of Tangents).** In a neutral geometry, if $\mathcal{C}$ is a circle and $Q \in \mathcal{C}$ then there is a unique line $t$ which is tangent to $\mathcal{C}$ and whose point of tangency is $Q$.

**10.** Prove the above Corollary. [Cor 6.5.7, p155]

**Definition**. **(continuous).** Function $h : \mathbb{R} \to \mathbb{R}$ is continuous at $t_0 \in \mathbb{R}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|h(t) - h(t_0)| \leq \varepsilon$ if $|t - t_0| < \delta$. (Thus if $t$ is "near" $t_0$ then $h(t)$ is "near" $h(t_0)$).

**Intermediate Value Theorem.** If $h : [a, b] \to \mathbb{R}$ is continuous at every $t_0 \in [a, b]$ and if $y$ is a number between $h(a)$ and $h(b)$ then there is a point $s \in [a, b]$ with $h(s) = y$.

**Theorem**. Let $r$ be a positive real number and let $A$, $B$, $C$ be points in a neutral geometry such that $AC < r$ and $\overrightarrow{AB} \perp \overline{AC}$. Then there is a point $D \in \overrightarrow{AB}$ with $CD = r$.

**11.** Prove the above Theorem. [Th 6.5.8, p156]

**Theorem**. **(Line-Circle Theorem).** In a neutral geometry, if a line $\ell$ intersects the interior of a circle $\mathcal{C}$, then $\ell$ is a secant.

**12.** Prove the above Theorem. [Th 6.5.9, p157]

**Theorem**. **(External Tangent Theorem).** In a neutral geometry, if $\mathcal{C}$ is a circle and $P \in \text{ext}(\mathcal{C})$, then there are exactly two lines through $P$ tangent to $\mathcal{C}$.

**13.** Prove the above Theorem. [Th 6.5.10, p158]

**14.** In a neutral geometry, if $\mathcal{C}$ is a circle with $A \in \text{int}(\mathcal{C})$ and $B \in \text{ext}(\mathcal{C})$, prove that $\overline{AB} \cap \mathcal{C} \neq \emptyset$.

# 21 The Two Circle Theorem

From previus lesson we know that two distinct circles in a neutral geometry intersect in at most two points. The main point of this section is to give a condition for when two circles intersect in exactly two points. This result, called the Two Circle Theorem, will follow directly from a converse of the Triangle Inequality.

**Theorem**. **(Sloping Ladder Theorem).** In a neutral geometry with right triangles $\triangle ABC$ and $\triangle DEF$ whose right angles are at $C$ and $F$, if $\overline{AB} \cong \overline{DE}$ and $\overline{AC} > \overline{DF}$, then $\overline{BC} < \overline{EF}$.

**1.** Prove the above Theorem. [Th 6.6.1, p160]

**Theorem**. Let $\overline{AB}$ and $\overline{DE}$ be two chords of the circle $\mathcal{C} = \mathcal{C}_r(C)$ in a neutral geometry. If $\overline{AB}$ and $\overline{DE}$ are both perpendicular to a diameter of $\mathcal{C}$ at points $P$ and $Q$ with $C - P - Q$, then $DQ < AP < r$.

**2.** Prove the above Theorem.

**Theorem**. **(Triangle Construction Theorem).** Let $\{\mathcal{S}, \mathcal{L}, d, m\}$ be a neutral qeometry and let $a$, $b$, $c$ be three positive numbers such that the sum of any two is greater than the third. Then there is a triangle in $\mathcal{S}$ whose sides have length $a$, $b$ and $c$.

**3.** Prove the above Theorem. [Th 6.6.3, p161]

**Theorem**. **(Two Circle Theorem).** In a neutral geometry, if $\mathcal{C}_1 = \mathcal{C}_b(A)$, $\mathcal{C}_2 = \mathcal{C}_a(B)$, $AB = c$, and if each of $a$, $b$, $c$ is less than the sum of the other two, then $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in exactly two points, and these points are on opposite sides of $\overleftrightarrow{AB}$.

**4.** Prove the above Theorem.

**Theorem**. If a protractor geometry satisfies SSS and both the Triangle Inequality and the Two Circle Theorem with the neutral hypothesis omitted, then it also satisfies SAS and is a neutral geometry.

**5.** Prove the above Theorem. [Th 6.6.6, p164]

**6.** Prove that in a neutral geometry, two circles $\mathcal{C}_1$ and $\mathcal{C}_2$intersect in exactly two points if and only $\mathcal{C}_1 \cap \text{int}(\mathcal{C}_2) \neq \emptyset$ and $\mathcal{C}_1 \cap \text{ext}(\mathcal{C}_2) \neq \emptyset$.

**7.** Prove that in a neutral geometry a circle of radius $r$ has a chord of length $c$ if and only if $0 < c \leq 2r$.

**8.** In a neutral geometry prove that for any $s > 0$ there is an equilateral triangle each of whose sides has length $s$.